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Corollary. **$\overline{A_{\mathrm{TM}}}$ is not Turing-recognizable, and thus, not decidable.**

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Corollary. **$\overline{A_{\mathrm{TM}}}$ is not Turing-recognizable, and thus, not decidable.**

For this corollary we need the following fact.

Fact. **A language L is decidable if and only if both L and $\overline{L}$ are Turing-recognizable.**

Proof of Corollary A_{TM} is Turing-recognizable and is not decidable. So, $\overline{A_{\mathrm{TM}}}$ is Turing-recognizable ▮ Corollary

Proof of Fact

Proof of Fact [$\Rightarrow$] Let L be decidable and let M be a Turing machine that decides L. By swapping q_{accept} and q_{reject} of M we get a Turing machine M' that decides $\overline{L}$. So both L and $\overline{L}$ are Turing-decidable, and thus, Turing-recognizable.

Proof of Fact (cont'd)

[$\Leftarrow$] Let L and $\overline{L}$ be recognized by TMs M_1 and M_2, respectively. Define a two-tape machine M that, on input x, does the following:

1. M copies x onto Tape 2.
2. M repeats the following until either M_1 or M_2 accepts:
   - M simulates one step of M_1 on Tape 1 then one step of M_2 on Tape 2.
3. M accepts x if either M_1 accepts x or M_2 rejects x; M rejects x if either M_2 accepts x or M_1 rejects x.

Then M decides L because for every x, at least one of of the two machines halts on input x. ▮ Fact

Diagonalization

A set S is **countable** if either it is finite or it has the same size as $\mathcal{N}$; i.e., there is a **one-to-one, onto correspondence** between S and $\mathcal{N}$ (or there is a **bijection** from S to $\mathcal{N}$).

Simple Facts About the Countable

Let $\mathcal{Q}$ be the set of all positive rational numbers and let $\mathcal{R}$ be the set of all positive real numbers.

Fact. **$\mathcal{Q}$ is countable while $\mathcal{R}$ is not.**

Proving the Fact

Proof Each member of $\mathcal{Q}$ is expressed as a fraction $\frac{m}{n}$ such that $m, n \in \mathcal{N}$ and $\gcd(m, n) = 1$.

So we have only to come up with a bijection from $\mathcal{N}$ to the set $\{\frac{m}{n} \mid m, n \geq 1 \& \gcd(m, n) = 1\}$.

$\mathcal{Q}$ is countable

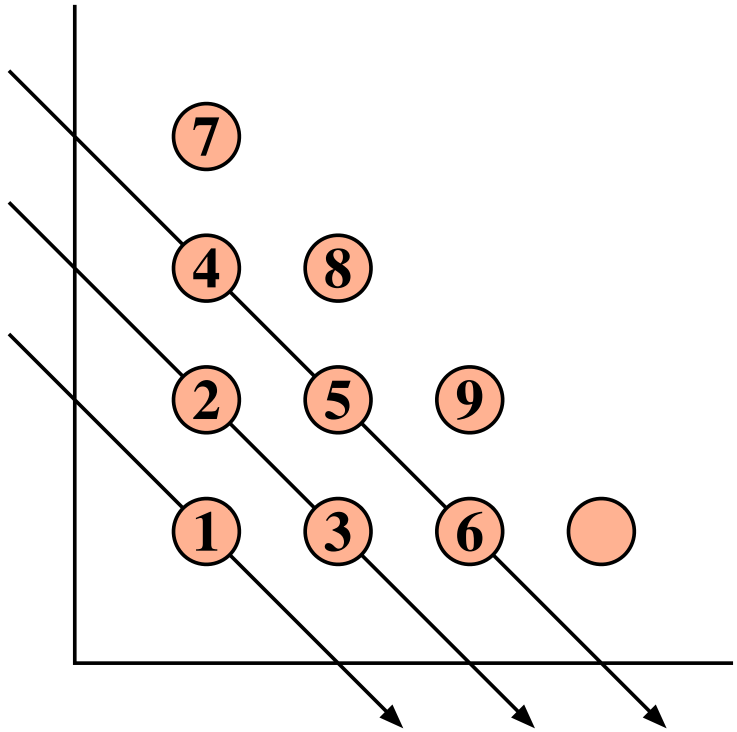
We will visit all the grid points in the first quadrant of the xy-plane.

For $p = 1, 2, 3, \ldots$, visit the points (x, y) on the line $x + y = p$

$$(1, p-1), (2, p-2), \ldots, (p-1, 1)$$

and collect only those points at which x and y are relatively prime to each other.

$\mathcal{Q}$ is countable



The numbers show the visiting order. Number 5 is $(2, 2)$ and thus is skipped.

$\mathcal{R}$ is not countable

Assume, by way of contradiction, that $\mathcal{R}$ is countable. Then the real numbers can be enumerated as $r_1, r_2, \ldots$.

Define x to be the number between 0 and 1 defined as follows:

(*) **For each $i \in \mathcal{N}$, the ith digit of x after the decimal point is that of r_i plus 1 (modulo 10).**

For example, if $r_1 = 3.\underline{1}4159, r_2 = 2.2\underline{3}606, r_3 = 1.73\underline{2}05, \ldots$, then $x = .243\ldots$,

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This x is real. By assumption there must exist a k such that r_k is x. However, by definition, the k-th digit of r_k is different from that of x, a contradiction.

Thus, $\mathcal{R}$ is not countable. ∎

An Immediate Application of Diagonalization

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Proof Consider all Turing machines whose input alphabet is $\{0\}$.

Since each Turing machine can be encoded as a word of finite length, this set of Turing machines is countable.

Let $M_1, M_2, \ldots$ be the enumeration of all Turing machines in this set.

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An Immediate Application of Diagonalization (cont'd)

Define $L = \{0^i \mid M_i$ on input 0^i does not accept $\}$.

There is no machine M_k that recognizes L. Why?

If there were such a k, then we have by definition of L

$$0^k \in L \Leftrightarrow M_k \text{ does not accept } 0^k.$$

However, the latter condition, by the definition of k, is equivalent to $0^k \notin L(M_k)$. Since $L(M_k) = L$, it is equivalent to $0^k \notin L$. Thus, we have

$$0^k \in L \Leftrightarrow 0^k \notin L,$$

a contradction. ∎

Proof of Theorem (A_{TM} is not decidable)

Assume that A_{TM} is decidable. Let T be a Turing machine that decides A_{TM}. Define D to be a machine that, on input w,

1. Check whether w is a legal encoding of some Turing machine, say M. If not, immediately reject w.
2. Simulate T on $\langle M, \langle M \rangle \rangle$.
3. If T accepts, then reject; otherwise, accept.

Since T decides A_{TM} by assumption, M always halts; so does D. For every Turing machine M,

$$D \textbf{ accepts } \langle M \rangle \Leftrightarrow M \textbf{ does not accept } \langle M \rangle$$

With $M = D$, we have

$$D \textbf{ accepts } \langle D \rangle \Leftrightarrow D \textbf{ does not accept } \langle D \rangle.$$

This is a contradiction. ∎